The Surface S

In this class, we will limit ourselves to studying only those surfaces that are formed when we change the location of a point by varying **two** coordinate parameters. In other words, the other coordinate parameters will remain **fixed**.

Mathematically, therefore, a **surface** is described by:

1 equality (e.g., $x = 2$ or $r = 3$)

AND

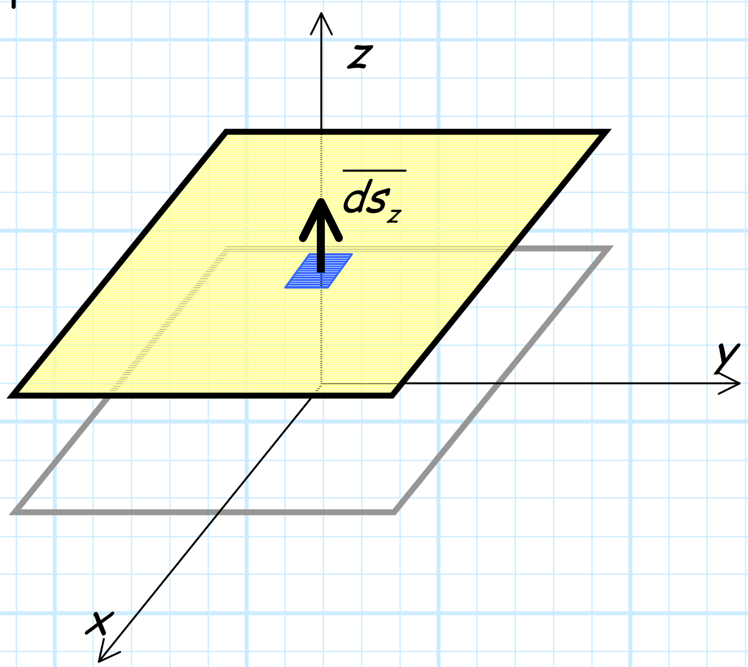
2 inequalities (e.g., $-1 < y < 5$ and $-2 < z < 7$, or $0 < \theta < \pi/2$ and $0 < \phi < \pi$)

Likewise, we will need to **explicitly** determine the **differential surface vector** $\overline{ds}$ for each contour.

We will be able to describe a surface for **each** of the coordinate values we have studied in this class!

Cartesian Coordinate Surfaces

The **single** equation $z = 3$ specifies **all** points $P(x,y,z)$ with a coordinate value $z = 3$. These points form a plane that is **parallel** to the x-y plane.



* As we move across this plane, the coordinate values of x and y **will** vary. Thus, the size of this **rectangular** plane is defined by **two inequalities** --
$c_{x1} \leq x \leq c_{x2}$ and $c_{y1} \leq y \leq c_{y2}$.

* Note the **differential surface vector** $\overline{ds_z}$ (or $-\overline{ds_z}$) is **orthogonal** to every point on this plane.

* Similarly, the equations $y = -2$ or $x = 6$ describe **planes** orthogonal to the x-z plane and the y-z plane, respectively. Likewise, the differential surface vectors $\overline{ds_y}$ and $\overline{ds_x}$ are orthogonal to each point on **these** planes.

Summarizing the **Cartesian surfaces**:

1. Flat **plane** parallel to the *y-z* plane.

$$x = c_x \qquad c_{y1} \leq y \leq c_{y2} \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_x} = \pm \hat{a}_x \, dy \, dz$$

2. Flat **plane** parallel to the *x-z* plane.

$$c_{x1} \leq x \leq c_{x2} \qquad y = c_y \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_y} = \pm \hat{a}_y \, dz \, dx$$

3. Flat **plane** parallel to the *x-y* plane.

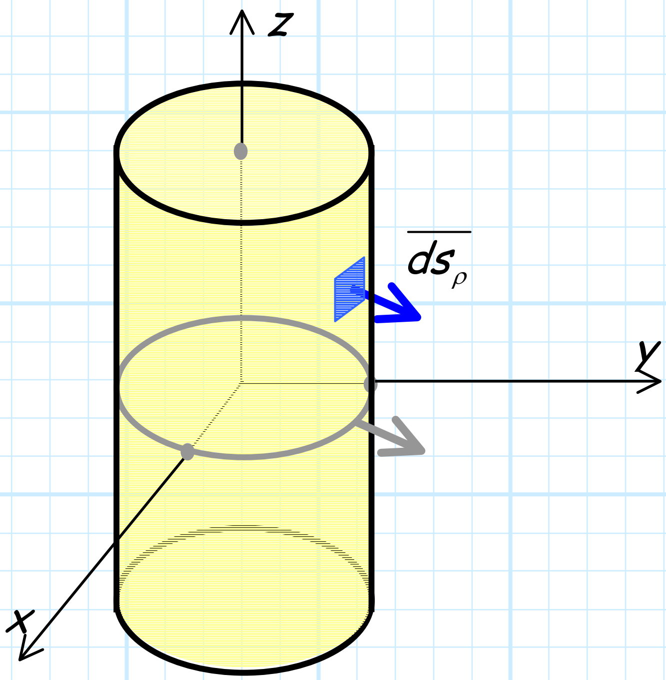
$$c_{x1} \leq x \leq c_{x2} \qquad c_{y1} \leq y \leq c_{y2} \qquad z = c_z$$

$$\overline{ds} = \pm \overline{ds_z} = \pm \hat{a}_z \, dy \, dx$$

Cylindrical Coordinate Surfaces

With **cylindrical** coordinates, we can define surfaces such as $\phi = 45^\circ$ or $\rho = 4$. These surfaces, however, are more **complex** than simply planes.

For example, the surface denoted by $\rho=4$ is formed by all points with coordinate $\rho=4$. In other words, this surface is formed by **all** points that are a distance of 4 units from the *z*-axis—a **cylinder** !

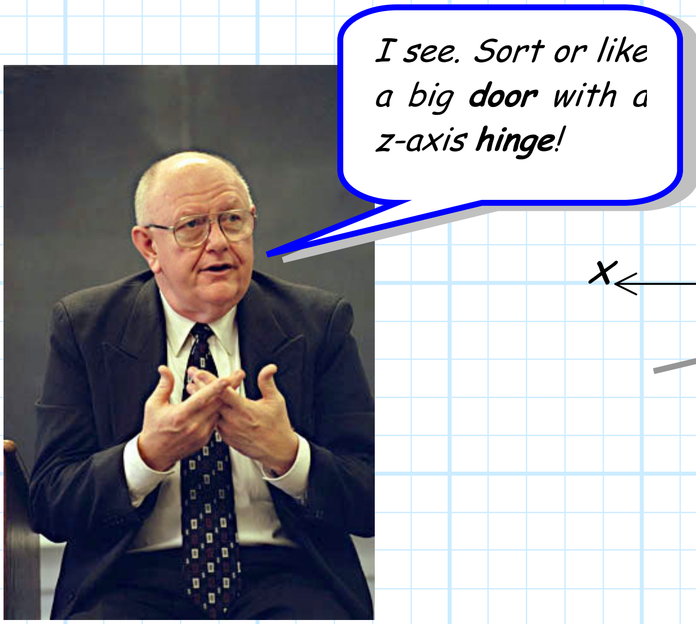


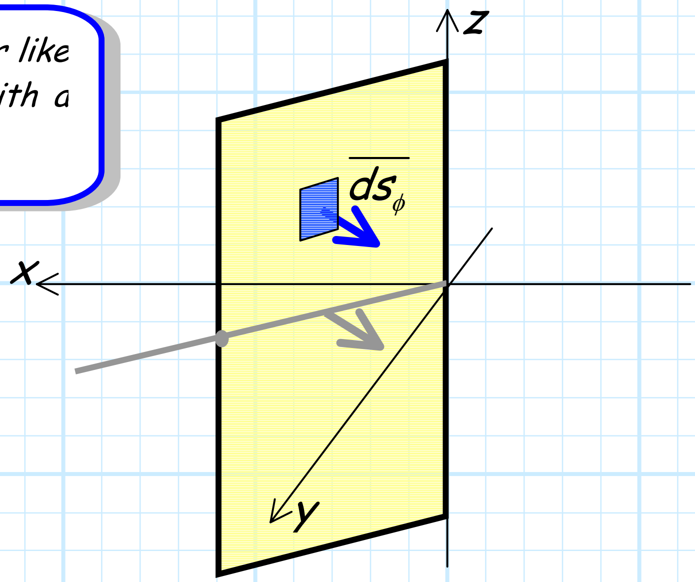
* As we move across this cylinder, the coordinate values of ϕ and *z* **will** vary. Thus, the size of this cylinder is defined by **two inequalities**--$c_{\phi 1} \leq \phi \leq c_{\phi 2}$ and $c_{z1} \leq z \leq c_{z2}$.

* Note a cylinder that **completely surrounds** the *z*-axis is described by the inequality $0 \leq \phi \leq 2\pi$. However, the cylinder does **not** have to be complete! For example, the inequality $0 \leq \phi \leq \pi$ defines a **half**-cylinder,

* We note the differential surface vector $\overline{ds_\rho}$ (or $-\overline{ds_\rho}$) is orthogonal to this surface at **all** points.

Another surface is defined by the equation $\phi = 45^\circ$. This surface is formed only from points with coordinate value $\phi = 45^\circ$. The surface is a **half-plane** that extends outward from the *z*-axis.



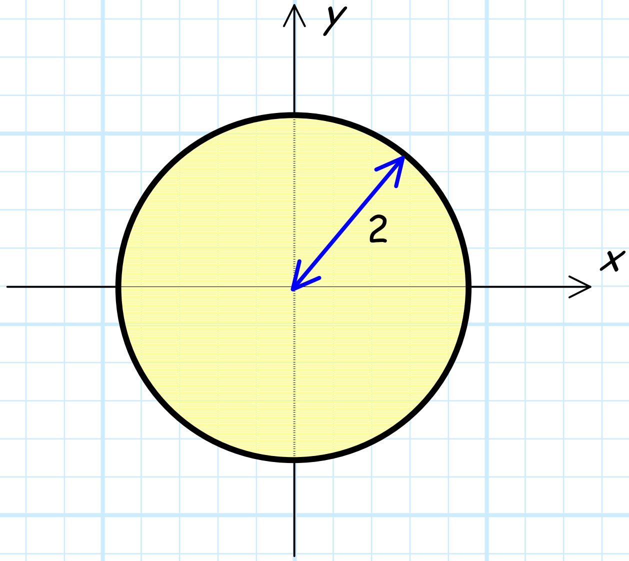


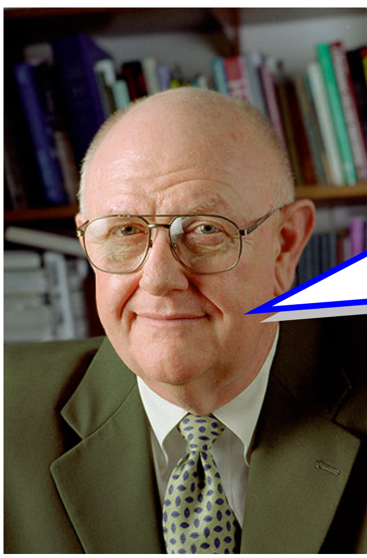
Note the differential surface vector $\overline{ds_\phi}$ is **orthogonal** to this surface at every point.

The **final cylindrical surface** that we will consider the type formed by the equality *z* = 2. We know that this forms a **flat plane** that is parallel to the *x-y* plane.

* Using the inequalities of **Cartesian** coordinates, this flat plane is rectangular in shape. However, using **cylindrical** coordinates inequalities, this plane will be shaped like a **ring** or a **disk**.

* For example, the surface $z = 0,\ 0 \le \rho \le 2,\ 0 \le \phi \le 2\pi$ describes a circular disk of radius 2, lying **on** the x-y plane, and centered at the **z**-axis:



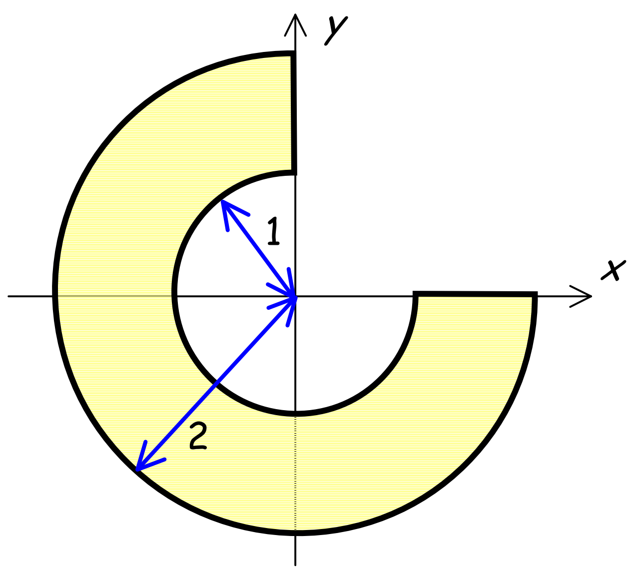


Now let's see if ***you've*** *been paying attention! Determine the two* ***inequalities*** *that define this flat surface.*

$z = 0$

$\leq \rho \leq$

$\leq \phi \leq$



Summarizing our **cylindrical surface** results:

1. Circular **cylinder** centered around the *z*-axis.

$$\rho = c_\rho \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\rho} = \pm \hat{a}_p \, \rho \, d\phi \, dz$$

2. "Vertical" **half-plane** extending from the *z*-axis.

$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad \phi = c_\phi \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\phi} = \pm \hat{a}_\phi \, dz \, d\rho$$

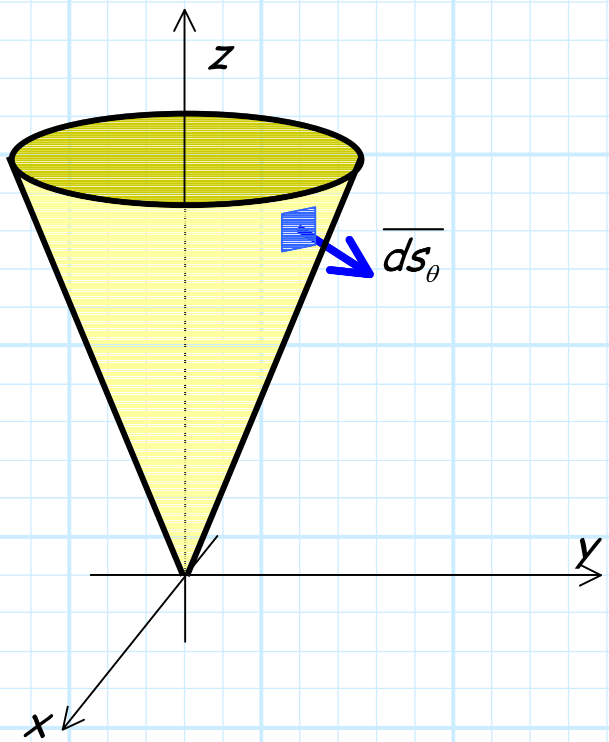
3. Flat **plane** parallel to the *x-y* plane.

$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad z = c_z$$

$$\overline{ds} = \pm \overline{ds_z} = \pm \hat{a}_z \, \rho \, d\phi \, d\rho$$

Spherical Coordinate Surfaces

The surface defined by $\theta = 30^\circ$ is formed only from points with coordinate $\theta = 30^\circ$. This surface is a **cone**! The apex of the cone is centered at the origin, and its axis of rotation is the z-axis.



* Note that the differential surface vector $\overline{ds_\theta}$ is **normal** to this surface at every point.

* Just like a cylinder, a **complete** cone is defined by the inequality $0 \le \phi \le 2\pi$. Alternatively, for example, the equation $\pi \le \phi \le 3\pi/2$ defines a **quarter** cone.

Say instead our equality equation is $r=3$. This defines a surface formed from all points a distance of 3 units from the origin—a **sphere** of radius 3 !

* This sphere is **centered** at the origin.

* The differential surface vector $\overline{ds_r}$ is normal to this sphere at all points on the surface.

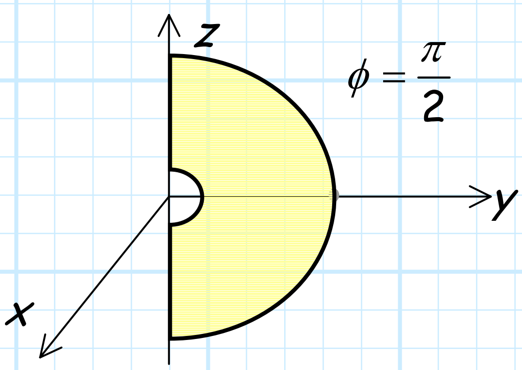
* If we wish to define a **complete** sphere, our inequalities must be:

$$0 \leq \theta < \pi \qquad \text{and} \qquad 0 \leq \phi < 2\pi$$

otherwise, we will be defining some **subsection** of a spherical surface (e.g., the "Northern Hemisphere".).

Finally, we know that the equation $\phi = 45^\circ$ defines a vertical **half-plane**, extending from the z-axis.

However, using **spherical** inequalities, this vertical plane will be in the shape of a **semi-circle** (or some section thereof), as opposed to rectangular (with cylindrical inequalities).



Summarizing the **spherical surfaces**:

1. **Sphere** centered at the origin.

$$r = c_r \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

$$\overline{ds} = \pm \overline{ds_r} = \pm \hat{a}_r \, r^2 \sin\theta \, d\theta \, d\phi$$

2. A **cone** with apex at the origin and aligned with the z-axis.

$$c_{r1} \leq r \leq c_{r2} \qquad \theta = c_\theta \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

$$\overline{ds} = \pm \overline{ds_\theta} = \pm \hat{a}_\theta \, r \sin\theta \, d\phi \, dr$$

3. "Vertical" **half-plane** extending **from** the z-axis.

$$c_{r1} \leq r \leq c_{r2} \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad \phi = c_\phi$$

$$\overline{ds} = \pm \overline{ds_\phi} = \pm \hat{a}_\phi \, r \, dr \, d\theta$$